

The Shannon-McMillan Theorem for AF C^* -systems

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Abstract

We give a new proof of quantum Shannon-McMillan theorem, extending it to AF C^* -systems. Our proof is based on the variational principle, instead of the classical Shannon-McMillan theorem.

1 Introduction

The classical Shannon-McMillan Theorem [S] states that an ergodic system has *typical sets* satisfying the asymptotic equipartition property. This theorem demonstrates the significance of the entropy which gives the *size* of the typical sets. There has recently been great progress in the quantum version of the Shannon-McMillan Theorem [HP2], [NS1], [BKSS], [BS]. In particular, Bjelaković et al. [BKSS] proved Shannon-McMillan theorem for ergodic quantum spin systems. The analyses in [HP2], [NS1], [BKSS], [BS] are based on the classical Shannon-McMillan theorem. The quantum mean entropy of a translation invariant state can be approximated by classical mean entropies of its restriction to some abelian subalgebras. This fact enables us to reduce the problem to the classical one.

In this paper, we present an alternative proof of quantum Shannon-McMillan theorem. Our proof is based on the variational principle, which is a well-known thermodynamic property of quantum spin systems. Roughly speaking, the variational principle enables us to estimate *rank* of support projections of ergodic states, in terms of the mean entropy. By virtue of this estimate, we are able to prove quantum Shannon-McMillan theorem directly, without relying on the classical version of it. Using this argument, we extend the quantum Shannon-McMillan theorem to AF C^* -systems. Our proof applies to any dynamical system which admits thermodynamical formalism. In particular, we can apply it to quantum spin systems on \mathbb{Z}^ν -lattice with $\nu \geq 2$. However, for the sake of simplicity, in this paper, we present the result for \mathbb{Z} -action.

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We consider quadruples $(A, \{A_{[n,m]}\}_{n \leq m}, \tau, \gamma)$, $n, m \in \mathbb{Z}$, where A is a unital C^* -algebra, $A_{[n,m]}$ a finite-dimensional C^* -subalgebra of A with $1_A \in A_{[n,m]}$, τ a faithful trace state of A , and γ a τ -preserving automorphism of A . For any subset Λ of \mathbb{Z} , we denote by A_Λ the C^* -subalgebra of A generated by $A_{[k,n]}, [k, n] \subset \Lambda$, and write A_n instead of $A_{[0, n-1]}$. We denote the set of all self-adjoint elements of A by A_{sa} and the set of all γ -invariant states on A by \mathcal{S}_γ . For a family of intervals $\{I_\alpha\}_\alpha$ in \mathbb{Z} , we will write $\bigvee_\alpha A_{I_\alpha}$ for the C^* -subalgebra generated by $\{A_{I_\alpha}\}_\alpha$. If B is a finite-dimensional C^* -subalgebra of A , its canonical trace is denoted by Tr_B . The restriction of a state φ on A to B is written $\varphi|_B$. For a state ψ on B , we denote its density matrix by D_ψ , and von Neumann entropy by $S(\psi)$. For positive linear functionals ψ_1, ψ_2 on B , the relative entropy is denoted by $S(\psi_1, \psi_2)$. Throughout the paper we suppose that the following conditions are satisfied.

- Assumption 1.1** (i) *There exists $n_0 > 0$ such that $A_{(-\infty, 0]}$ and $A_{[n_0, \infty)}$ commute;*
- (ii) $\tau(ab) = \tau(a)\tau(b)$ for $a \in A_{(-\infty, 0]}$, $b \in A_{[n_0, \infty)}$;
- (iii) $A_{[n', m']} \subset A_{[n'', m'']}$, for $n'' \leq n' \leq m' \leq m''$;
- (iv) $\bigcup_n A_{[-n, n]}$ is dense in A ;
- (v) $\gamma(A_{[n, m]}) = A_{[n+1, m+1]}$ for all $n \leq m$, $n, m \in \mathbb{Z}$;
- (vi) *for the density matrix $D_{\tau|_{A_n}}$ the limit $\lambda_\tau := \lim_{n \rightarrow \infty} -\frac{1}{n} \log D_{\tau|_{A_n}}$ exists in norm, and is a scalar.*

This class of systems were studied in [HP1], [GN]. Clearly, quantum spin chains belong to the class. See [HP1], [GN] for the other examples.

For such a system, the mean entropy exists.

Proposition 1.2 *For any γ -invariant state ω , the limit*

$$s(\omega) := \lim_n \frac{1}{n} S(\omega|_{A_n})$$

exists in $[0, \lambda_\tau]$. The function $\mathcal{S}_\gamma \ni \omega \mapsto s(\omega) \in [0, \lambda_\tau]$ is affine and weakly upper semicontinuous.*

This system also satisfies the variational principle.

Proposition 1.3 *Let a be a self-adjoint element in A . Then the limit*

$$P_\gamma^\tau(a) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \tau(e^{-\sum_{i=0}^{n-1} \gamma_i(a)})$$

exists and

$$P_\gamma^\tau(a) = \sup_{\omega \in \mathcal{S}_\gamma} \{s(\omega) - \omega(a)\} - \lambda_\tau.$$

Proposition 1.2 and 1.3 are well-known for quantum spin systems. (See [BR2]). For AF C^* -systems, $n_0 = 1$ case was shown in [HP1]. We give a proof for the general case in Section 3.

In this paper, we give a proof of quantum Shannon-McMillan Theorem in AF C^* -systems via the variational principle, without relying on the classical Shannon-McMillan Theorem.

Theorem 1.4 *Let $(A, \{A_{[n,m]}\}_{n \leq m}, \tau, \gamma)$ be an AF C^* -system satisfying Assumption 1.1. Then for any γ -ergodic state ω on A and $\delta > 0$, there exists a sequence of projections $\{p_n\}_{n \in \mathbb{N}}$ in A with $p_n \in A_n$, such that*

(i) $\lim_{n \rightarrow \infty} \omega(p_n) = 1$;

(ii) for all minimal projections $e \in A_n$ with $e \leq p_n$,

$$e^{-n(s(\omega)+\delta)} \leq \omega(e) \leq e^{-n(s(\omega)-\delta)};$$

(iii) for n large enough,

$$e^{n(s(\omega)-\delta)} \leq \text{Tr}_{A_n} p_n \leq e^{n(s(\omega)+\delta)}.$$

2 Shannon-McMillan Theorem for AF C^* -systems

For a self-adjoint element a in a finite dimensional C^* -algebra and a Borel set I of \mathbb{R} , $\text{Proj}[a \in I]$ denotes the spectral projection of a associated with I .

Lemma 2.1 *For any γ -ergodic state ω and any*

$$t < -s(\omega) + \lambda_\tau, \tag{1}$$

we have

$$\lim_{n \rightarrow \infty} \omega \left(\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq t \right] \right) = 0.$$

Proof. By Proposition 1.3 and concavity and upper semi-continuity of the mean entropy (Proposition 1.2), we have

$$s(\omega) = \inf_{a \in A_{sa}} \{P_\gamma^\tau(a) + \omega(a) + \lambda_\tau\}.$$

(See Theorem 3.12 of [R].) Therefore, (1) means there exists a self-adjoint element $a \in A$ such that

$$t < -(P_\gamma^\tau(a) + \omega(a)).$$

Since the right hand side is continuous with respect to the norm of a and invariant under the translation $a \rightarrow \gamma_i(a)$, we may assume a is in A_l for some $l \in \mathbb{N}$. For each $l \leq n \in \mathbb{N}$, we set $t_n(a) := \sum_{i=0}^{n-l} \gamma_i(a)/n \in A_n$.

Take $0 < \delta < -t - (P_\gamma^\tau(a) + \omega(a))$, and define

$$F^n := \text{Proj}[t_n(a) \in (\omega(a) - \delta, \omega(a) + \delta)] \in A_n$$

for each $n \in \mathbb{N}$ with $l \leq n$.

By the ergodicity of ω , we have

$$\lim_{n \rightarrow \infty} \omega(F^n) = 1. \quad (2)$$

(See Theorem 4.3.17 of [BR1].) We claim

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau(F^n) \leq P_\gamma^\tau(a) + \omega(a) + \delta. \quad (3)$$

This is trivial if $F^n = 0$ eventually. By taking a subsequence if necessary, we may assume $F^n \neq 0$ for all n . Then $\tau(F^n \cdot) / \tau(F^n)|_{A_n}$ and $\tau(e^{-nt_n(a)} \cdot) / \tau(e^{-nt_n(a)})|_{A_n}$ are states on A_n . By the positivity of relative entropy, we have

$$\begin{aligned} 0 &\leq S\left(\frac{\tau(F^n \cdot)}{\tau(F^n)}|_{A_n}, \frac{\tau(e^{-nt_n(a)} \cdot)}{\tau(e^{-nt_n(a)})}|_{A_n}\right) \\ &= -\log \tau(F^n) + n \frac{\tau(F^n t_n(a))}{\tau(F^n)} + \log \tau(e^{-nt_n(a)}) \\ &\leq -\log \tau(F^n) + n(\omega(a) + \delta) + \log \tau(e^{-nt_n(a)}). \end{aligned}$$

This immediately implies (3).

Next for each n , we construct a projection Q^n commuting with the density matrix $D_{\omega|_{A_n}}$ and satisfying $\tau(Q^n) = \tau(F^n)$ and $\omega(F^n) \leq \omega(Q^n)$. Write A_n in the form

$$A_n = \bigoplus_{k=1}^{N_n} M_k^n,$$

where M_k^n is isomorphic to a matrix algebra $\text{Mat}_{m_k^n}(\mathbb{C})$. Let z_k^n be the central projection in A_n such that $M_k^n = A_n z_k^n$. Since τ is trace, the density matrix $D_{\tau|_{A_n}}$ of $\tau|_{A_n}$ has the form

$$D_{\tau|_{A_n}} = \bigoplus_{k=1}^{N_n} \lambda_k^n z_k^n$$

with $0 \leq \lambda_k^n \leq 1$. Write $D_{\omega|_{A_n}}$ and F^n as

$$D_{\omega|_{A_n}} = \bigoplus_{k=1}^{N_n} D_k^n, \quad F^n = \bigoplus_{k=1}^{N_n} F_k^n$$

with positive elements $D_k^n \in M_k^n$ and projections $F_k^n \in M_k^n$.

For each k , write D_k^n in the form $D_k^n = \sum_{i=1}^{m_k^n} \beta_{i,k}^n q_{i,k}^n$ where $\beta_{i,k}^n \geq 0$ and $q_{i,k}^n$ are mutually orthogonal minimal projections in A_n with $\sum_{i=1}^{m_k^n} q_{i,k}^n = z_k^n$. We

may assume $\beta_{1,k}^n \geq \beta_{i,k}^n \geq \cdots \geq \beta_{m_k^n,k}^n$. Set $Q_k^n := \sum_{i=1}^{\text{Tr}_{M_k^n} F_k^n} q_{i,k}^n$. Then, by Ky Fan's Theorem,

$$\text{Tr}_{M_k^n}(D_k^n Q_k^n) \geq \text{Tr}_{M_k^n}(D_k^n F_k^n), \quad \text{Tr}_{M_k^n} Q_k^n = \text{Tr}_{M_k^n} F_k^n.$$

The projection $Q^n := \bigoplus_{k=1}^{N_n} Q_k^n$ satisfies the required condition.

To complete the proof, note that Q^n and $\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq t \right]$ commute. Therefore, we have

$$\begin{aligned} & \omega \left(\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq t \right] \right) \\ &= \omega \left(\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq t \right] Q^n \right) \\ &+ \omega \left(\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq t \right] (1 - Q^n) \right) \\ &\leq e^{nt} \tau(Q^n) + \omega(1 - Q^n). \end{aligned}$$

It suffices to show that each term on the right hand side converges to 0 as $n \rightarrow \infty$. From $\tau(Q^n) = \tau(F^n)$ and (3), we have the exponentially fast decay of the first term:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log e^{nt} \tau(Q^n) &= t + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau(Q^n) \\ &= t + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \tau(F^n) \leq t + P_\gamma^\tau(a) + \omega(a) + \delta < 0. \end{aligned}$$

Furthermore, $\omega(F^n) \leq \omega(Q^n)$ and (2) implies $\lim_{n \rightarrow \infty} \omega(1 - Q^n) = 0$. \square

Proof of Theorem 1.4. From the condition (vi) in Assumption 1.1, it suffices to show that for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \omega \left(\text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \in (-s(\omega) + \lambda_\tau - \delta, -s(\omega) + \lambda_\tau + \delta) \right] \right) = 1. \quad (4)$$

Theorem 1.4 follows from this with

$$\begin{aligned} p_n &:= \text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \in (-s(\omega) + \lambda_\tau - \delta/3, -s(\omega) + \lambda_\tau + \delta/3) \right] \\ &\cdot \text{Proj} \left[\frac{1}{n} \log D_{\tau|_{A_n}} \in (-\lambda_\tau - \delta/3, -\lambda_\tau + \delta/3) \right]. \end{aligned}$$

As we already have Lemma 2.1, in order to show (4), it suffices to show that for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \omega \left(\text{Proj} \left[\frac{1}{n} \log (D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \geq -s(\omega) + \lambda_\tau + \delta \right] \right) = 0.$$

Fix $\varepsilon > 0$ and choose $\delta' > 0$ with $\delta'/(\delta' + \delta) < \varepsilon$. Set

$$\begin{aligned} Q_n^- &:= \text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \leq -s(\omega) + \lambda_\tau - \delta' \right], \\ Q_n^+ &:= \text{Proj} \left[\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \geq -s(\omega) + \lambda_\tau + \delta \right]. \end{aligned}$$

By the positivity of relative entropy $S(\omega(Q_n^- \cdot)/\omega(Q_n^-)|_{A_n}, \tau(Q_n^- \cdot)/\tau(Q_n^-)|_{A_n})$ and the condition (vi) in Assumption (1.1) we have

$$\frac{1}{n} \omega(Q_n^-) \log \omega(Q_n^-) \leq \frac{1}{n} \omega(Q_n^- (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}})) \leq C \omega(Q_n^-),$$

for n large enough, with some positive constant C . As we have $\lim_{n \rightarrow \infty} \omega(Q_n^-) = 0$ from Lemma 2.1, these inequalities implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \omega(Q_n^- (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}})) = 0. \quad (5)$$

By the definitions of Q_n^- , Q_n^+ , we have

$$\begin{aligned} &\omega \left(\frac{1}{n} (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}}) \right) \\ &\geq \frac{1}{n} \omega(Q_n^- (\log D_{\omega|_{A_n}} - \log D_{\tau|_{A_n}})) \\ &\quad + (\omega(Q_n^-) - 1) (s(\omega) - \lambda_\tau + \delta') + (\delta + \delta') \omega(Q_n^+). \end{aligned}$$

Taking $n \rightarrow \infty$ limit, we obtain

$$\limsup_n \omega(Q_n^+) \leq \frac{\delta'}{\delta' + \delta} < \varepsilon.$$

□

3 Variational principle

In this section, we give a proof of Proposition 1.2 and 1.3. Most of the arguments are analogous to those in quantum spin systems. By the condition (vi) of Assumption 1.1, in order to prove Proposition 1.2 it suffices to consider

$$-\frac{1}{n} S(\omega|_{A_n}, \tau|_{A_n}) = \frac{1}{n} S(\omega|_{A_n}) + \frac{1}{n} \omega(\log D_{\tau|_{A_n}})$$

instead of $\frac{1}{n} S(\omega|_{A_n})$.

Fix some $m \in \mathbb{N}$. Set $I_{m,j} := [(m+n_0)j, (m+n_0)(j+1) - n_0 - 1]$ for each $j \in \mathbb{Z}$. From (i) of Assumption 1.1, there exists a $*$ -homomorphism $\pi_m : \bigotimes_{\mathbb{Z}} A_m \rightarrow$

$\bigvee_{j \in \mathbb{Z}} A_{I_m, j}$ with $\pi_m|_{\bigotimes_{j=k}^l A_m} (\bigotimes_{j=k}^l b_j) = \prod_{j=k}^l \gamma^{(m+n_0)j}(b_j)$, $b_j \in A_m$ for $j = k, \dots, l$, $k \leq l$. Furthermore, (ii) of Assumption 1.1 implies $\tau \circ \pi_m = \bigotimes_{\mathbb{Z}} \tau|_{A_m}$. From this π_m is $*$ -isomorphism.

Proof of Proposition 1.2. For each $n \geq m + n_0 + 1$, we have

$$\begin{aligned} -S(\omega|_{A_n}, \tau|_{A_n}) &\leq -S\left(\omega|_{\bigvee_{j=0}^{\lfloor \frac{n}{m+n_0} \rfloor - 1} A_{I_m, j}}, \tau|_{\bigvee_{j=0}^{\lfloor \frac{n}{m+n_0} \rfloor - 1} A_{I_m, j}}\right) \\ &= -S\left(\omega \circ \pi_m|_{\bigotimes_{j=0}^{\lfloor \frac{n}{m+n_0} \rfloor - 1} A_m}, \tau \circ \pi_m|_{\bigotimes_{j=0}^{\lfloor \frac{n}{m+n_0} \rfloor - 1} A_m}\right) \leq -\left[\frac{n}{m+n_0}\right] S(\omega|_{A_m}, \tau|_{A_m}). \end{aligned}$$

We used monotonicity of relative entropy in the first inequality and the subadditivity of von Neumann entropy in the second inequality. This inequality and Lemma 1.1.2 of [NS2] imply the existence of the limit $\lim_{n \rightarrow \infty} -\frac{1}{n} S(\omega|_{A_n}, \tau|_{A_n}) = \inf_m -S(\omega|_{A_m}, \tau|_{A_m})/(m+n_0)$. The rest of the proposition is standard. We leave it to the reader to check it. (See [BR2], [HP1].) \square

Lemma 3.1 *For any self-adjoint element a in A , the limit*

$$P_\gamma^\tau(a) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \tau(e^{-\sum_{i=0}^{n-1} \gamma_i(a)})$$

exists. The function P_γ^τ is continuous with respect to the norm topology on A_{sa} .

Proof. From Corollary 2.3.11 of [NS2], we may assume that $a \in A_l$ for some $l \in \mathbb{N}$. Fix some $m \in \mathbb{N}$, with $l \leq m$. Then from the condition (i), (ii) of Assumption 1.1, and again Corollary 2.3.11 of [NS2], we have

$$\begin{aligned} &\log \tau\left(e^{-\sum_{i=0}^{n-1} \gamma_i(a)}\right) \\ &\leq \left[\frac{n}{m+n_0}\right] \log \tau\left(e^{-\sum_{i=0}^{m-l} \gamma_i(a)}\right) + \left[\frac{n}{m+n_0}\right] (n_0 + l - 1) \|a\| + (m+n_0) \|a\|, \end{aligned}$$

for all $m+n_0 \leq n \in \mathbb{N}$. This inequality and Lemma 1.1.2 of [NS2] implies the existence of $P_\gamma^\tau(a)$. Continuity of P_γ^τ follows from Corollary 2.3.11 of [NS2]. (See [HP1].) \square

Proof of Proposition 1.3. The inequality

$$P_\gamma^\tau(a) \geq \sup_{\omega \in \mathcal{S}_\gamma} \{s(\omega) - \omega(a)\} - \lambda_\tau$$

follows from positivity of relative entropy $S(\omega|_{A_n}, \tau(e^{-\sum_{i=0}^{n-l} \gamma_i(a)})) / \tau(e^{-\sum_{i=0}^{n-l} \gamma_i(a)})|_{A_n}$.

Since P_γ^τ is continuous, to prove the converse inequality we may assume $a \in A_l$ for some $l \in \mathbb{N}$. Fix $l \leq m \in \mathbb{N}$. From the condition (i), (ii) of Assumption 1.1, there exists a γ_{m+n_0} -invariant state $\varphi_{m,a}$ given by

$$\varphi_{m,a} = \text{wk}^* - \lim_{N \rightarrow \infty} \frac{\tau \left(e^{-\sum_{j=-N}^N \sum_{i=0}^{m-l} \gamma_{j(m+n_0)+i}(a)} \right)}{\tau \left(e^{-\sum_{j=-N}^N \sum_{i=0}^{m-l} \gamma_{j(m+n_0)+i}(a)} \right)}.$$

The restriction of $\varphi_{m,a}$ to $A_{[k(m+n_0), n(m+n_0)+m]}$ with $k \leq n$, $k, n \in \mathbb{Z}$ is

$$\varphi_{m,a}|_{A_{[k(m+n_0), n(m+n_0)+m]}} = \frac{\tau \left(e^{-\sum_{j=k}^n \sum_{i=0}^{m-l} \gamma_{j(m+n_0)+i}(a)} \right)}{\tau \left(e^{-\sum_{j=k}^n \sum_{i=0}^{m-l} \gamma_{j(m+n_0)+i}(a)} \right)} \Big|_{A_{[k(m+n_0), n(m+n_0)+m]}}.$$

By monotonicity and positivity of relative entropy, we have

$$\begin{aligned} 0 &\leq S \left(\varphi_{m,a} \circ \gamma_i|_{A_{[0, n(m+n_0)-1]}}, \frac{\tau \left(e^{-\sum_{k=0}^{n(m+n_0)-l-1} \gamma_k(a)} \right)}{\tau \left(e^{-\sum_{k=0}^{n(m+n_0)-l-1} \gamma_k(a)} \right)} \Big|_{A_{[0, n(m+n_0)-1]}} \right) \\ &\leq S \left(\varphi_{m,a} \circ \gamma_i|_{A_{[-i, (n+1)(m+n_0)+m-i]}}, \frac{\tau \left(e^{-\sum_{k=0}^{n(m+n_0)-l-1} \gamma_k(a)} \right)}{\tau \left(e^{-\sum_{k=0}^{n(m+n_0)-l-1} \gamma_k(a)} \right)} \Big|_{A_{[-i, (n+1)(m+n_0)+m-i]}} \right), \end{aligned}$$

for each $i = 0, \dots, m+n_0-1$ and $n \in \mathbb{N}$. From this we get

$$\begin{aligned} 0 &\leq -S(\varphi_{m,a} \circ \gamma_i|_{A_{n(m+n_0)}}) - \varphi_{m,a} \circ \gamma_i \left(\log D_{\tau|_{A_{n(m+n_0)}}} \right) \\ &\quad + \log \tau \left(e^{-\sum_{k=0}^{n(m+n_0)-l-1} \gamma_k(a)} \right) + \varphi_{m,a} \left(\sum_{k=i}^{n(m+n_0)-l+i-1} \gamma_k(a) \right) \\ &\leq 2 \|a\| (4(m+n_0) + n(n_0+l)), \end{aligned} \tag{6}$$

for each $i = 0, \dots, m+n_0-1$ and $n \in \mathbb{N}$.

Note that for a γ_{m+n_0} -invariant state φ , the mean entropy with respect to γ_{m+n_0} exists:

$$s^{m+n_0}(\varphi) := \lim_{n \rightarrow \infty} \frac{1}{n} S(\varphi|_{A_{n(m+n_0)}}).$$

This s^{m+n_0} is also affine on the set of all γ_{m+n_0} -invariant states. If φ is γ -invariant, we have $s^{m+n_0}(\varphi) = (m+n_0)s(\varphi)$.

From (6), we have

$$s^{m+n_0}(\varphi_{m,a} \circ \gamma_i) - (m+n_0)\lambda_\tau \geq (m+n_0)P_\gamma^\tau(a) + \sum_{k=0}^{m+n_0-1} \varphi_{m,a} \circ \gamma_k(a) - 2\|a\|(n_0+l). \tag{7}$$

Define a γ -invariant state $\psi_{m,a}$ by $\psi_{m,a} := \left(\sum_{i=0}^{m+n_0-1} \varphi_{m,a} \circ \gamma_i \right) / (m+n_0)$. As s^{m+n_0} is affine and $\psi_{m,a}$ is γ -invariant, (7) implies

$$\begin{aligned} s(\psi_{m,a}) &= \frac{1}{m+n_0} s^{m+n_0}(\psi_{m,a}) = \frac{1}{(m+n_0)^2} \sum_{i=0}^{m+n_0-1} s^{m+n_0}(\varphi_{m,a} \circ \gamma_i) \\ &\geq P_\gamma^\tau(a) + \lambda_\tau + \psi_{m,a}(a) - \frac{2\|a\|(n_0+l)}{m+n_0}. \end{aligned}$$

Hence we obtain the converse inequality

$$P_\gamma^\tau(a) \leq \sup_{\omega \in \mathcal{S}_\gamma} \{s(\omega) - \omega(a)\} - \lambda_\tau.$$

□

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